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LETTER TO THE EDITOR

Critical behaviour of a forest fire model with immune trees

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Abstract. A detailed study of the critical, subcritical and supercritical behaviour of a forest fire model with immune trees is presented and it is demonstrated that the model belongs to the same universality class as Reggeon field theory. Consequently, problems emerging from (apparently) unrelated branches of science such as particle physics, catalysis, directed percolation and epidemic spreading can be understood by means of a unified description.

Irreversible dynamical systems are relevant to a wide range of phenomena in physics, chemistry, biology, ecology, etc. Within this context, forest fire models, introduced to describe the spreading of an epidemic disease [1, 2], have, very recently, become the focus of increasing interest [3–10]. Forest fire models are defined on d-dimensional hypercubic lattices of linear size L. Each lattice site is either empty, a green tree or a burning tree. At each time step, the system is updated in parallel according to the following rules.

(i) A burning tree becomes an empty site.

(ii) Trees grow with probability p from empty sites.

(iii) A green tree becomes a burning tree with probability $(1-g)$ if at least one next neighbour is burning.

(iv) A green tree becomes a burning tree with probability $f \ll 1$ if no neighbour is burning.

Taking $g = 0$ in (iii) and neglecting rule (iv), one has the model of Bak *et al* [3] which is non-critical and exhibits a steady state, which is a succession of fire fronts with fractal dimension $D = 1$ [4]. A modification of this model is obtained by introducing the immunity g ($g < 1$ in (iii)) which is the probability that a tree is not ignited although one of its neighbours is burning [7]. This forest fire model with immune trees (FFMIT) exhibits interesting fluctuating percolation behaviour [7]. Another model, proposed by Drossel *et al* [5], takes $g = 0$ in (iii) and, in the limit $f \ll 1$, $f/p \to 0$, shows self-organized critical (SOC) behaviour in a non-conservative system [5, 6, 8, 9]. In a recent variant of the forest fire model, which also exhibits SOC, it is asumed that sparks are dropped at random and if they fall on a tree, the whole cluster of sites connected to it burns [10].

In this work, both the dynamic and static critical behaviour of the FFMIT are analysed in detail and it is unambiguously demonstrated that this model belongs to the universality class of Reggeon field theory. This finding shows the formal equivalence between the FFMIT and other problems emerging from different disciplines, such as for example, quantum particle physics [11, 12], directed percolation [13], irreversible catalytic systems [14, 15], pair contact processes [16], branching annihilating random walkers [17], etc.

Let us first qualitatively describe the behaviour of the FFMIT [7]. Scanning the $\{p, g\}$-plane, one has that, fixing an arbitrary grow probability and starting with a small immunity, increments of g causes the fire density of the steady state to decrease until the fire becomes irreversibly extinguished at a certain critical point at coordinates $\{p_c, g_c\}$. The set of critical points defines a critical curve $g_c(p)$. For $g \geqslant g_c(p)$, the final state of the system is a healthy forest without fire. So, in the $L = \infty$ limit, the critical curve $g_c(p)$ divides the $\{p, g\}$-plane into two regions: a steady state with fire fronts for $g < g_c(p)$ (i.e. the supercritical region) and a unique absorbing state with all sites occupied by green trees for $g \geqslant g_c(p)$ (i.e. the subcritical region).

It should be stressed that for finite L, the steady state is metastable because, due to fluctuations of the stochastic system, there is always a finite probability of the fire becoming irreversibly extinguished. This probability increases when approaching the critical curve and, consequently, it is not possible to obtain reliable data, by means of numerical simulations, in order to determine critical exponents. Also, the continuous transition between the stationary regime and the absorbing state is of second order, e.g. dominated by fluctuations, so a mean-field treatment is not adequate. These shortcomings can be avoided by evaluating critical exponents related to the dynamic critical behaviour of the system. For this purpose, one starts, at $t = 0$, with a small fire at the centre of the lattice otherwise filled by green trees, i.e. a configuration very close to the absorbing state. Then, the evolution of the fire is monitored and the following quantities are computed:

(i) the survival probability $P(t)$, i.e. the probability that the fire is still ignited at time t;
(ii) the average number of burning trees $N(t)$; and
(iii) the average mean-square distance $R^2(t)$ over which the fire has spread.

Notice that $N(t)$ is averaged over all samples, including those in which the fire has already been extinguished, while $R^2(t)$ is only averaged over samples which have burning trees. Averages are taking over 5×10^3 samples, and runs are performed up to $5 \times 10^2 \leqslant t \leqslant 10^3$. Simulations are performed in two dimensions and the lattice size is selected large enough, usuallly $L = 256$ or 512, in order to avoid the fire reaching the boundaries. Using this procedure, one can ensure that the data is free of undesired finite-size boundary effects. Close to the critical curve, the following scaling laws should hold [12]:

$$P(t) \propto t^{-\delta}\Phi\{\mathcal{D}(\Delta_1, \Delta_2)t^{1/\nu_\parallel}\} \qquad (1)$$

where $\Delta_1 = |g - g_c|$, $\Delta_2 = |p - p_c|$ and $\xi_t = \{\mathcal{D}(\Delta_1, \Delta_2)\}^{-\nu_\parallel}$† give the temporal correlations close to both g_c and p_c, $\nu_\parallel$ is the correlation length exponent (time direction) and δ is a critical exponent. Furthermore,

$$N(t) \propto t^{\eta}\varphi\{\mathcal{D}(\Delta_1, \Delta_2)t^{1/\nu_\parallel}\} \qquad (2)$$

and

$$R^2(t) \propto t^{z}\Psi\{\mathcal{D}(\Delta_1, \Delta_2)t^{1/\nu_\parallel}\} \qquad (3)$$

where both η and z are critical exponents. In the absorbing state, the correlations are short ranged and one therefore expects $P(t)$ and $N(t)$ to decay exponentially. This can only happen if

$$\varphi(x, y, t) \propto \{\mathcal{D}(x, y)t\}^{-\eta\nu_\parallel}\exp(-\{\mathcal{D}(x, y)t\}^{\nu_\parallel}) \qquad \text{for } x > 0,\ y > 0 \text{ and } t \to \infty.$$

† Notice that $\mathcal{D}(\Delta_1, \Delta_2)$ is the distance between the point of coordinates (g, p) and the critical point (p_g, g_c) at the critical curve, which may differ from the simple Euclidean distance $(\Delta_1^2 + \Delta_2^2)^{1/2}$. The same holds for $\Gamma(\Delta_1, \Delta_2)$ in equation (6).

Therefore, one obtains, from equation (2),

$$N(t) \propto \{\mathcal{D}(\Delta_1, \Delta_2)\}^{-\eta\nu_\parallel} \exp(-\{\mathcal{D}(\Delta_1, \Delta_2)\}^{\nu_\parallel} t) \qquad t \to \infty. \tag{4}$$

At criticality, one expects that log–log plots of $P(t)$, $N(t)$ and $R^2(t)$ would give straight lines, while upward and downward deviations would occur even slightly off-criticality. This behaviour would allow a precise determination of the critical points and critical exponents, as, for example, shown in figure 1. Table 1 summarizes the critical points and critical exponents obtained along the critical curve $g_c(p)$ using the method described above. The obtained exponents are in excellent agreement with those evaluated for directed percolation (DP) in (2+1) dimensions and the monomer–dimer catalytic model in 2 dimensions, as shown in table 1. This result strongly supports the Jannsen conjecture [19] that a continuous transition into an absorbing state characterized by a scalar order parameter may belong to the universality class of Reggeon field theory, or equivalently DP.

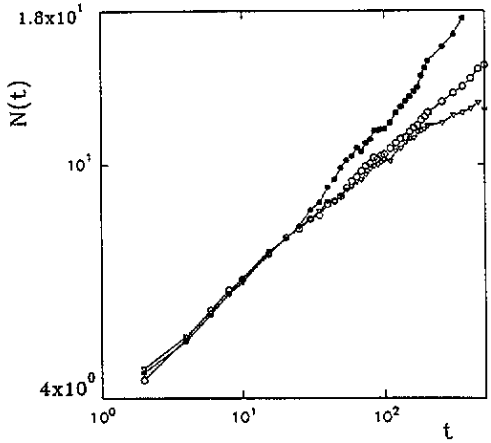

Figure 1. Log–log plots of $N(t)$ against t for $p = 0.1020$ and varying g: $g = 0.5000$ (supercritical) (●), $g = 0.5003$ (critical) (○) and $g = 0.5005$ (subcritical) (▽).

Table 1. Critical points and critical exponents for the FFMIT. Errors in the critical points and statistical errors in the exponents are the last digit. DP ≡ directed percolation in (2+1) dimensions. DMM ≡ dimer–monomer model as defined in [14].

	p_c	g_c	δ	η	z	$\nu_\parallel$	β	Reference
FFMIT	0.1020	0.5003	0.459	0.212	1.141	1.296	0.595[a]	PW
FFMIT	0.5000	0.5614	0.461	0.211	1.110	1.285	0.592[a]	PW
FFMIT	1.0000	0.5762	0.458	0.219	1.135	1.281	0.589[a]	PW
DP	—	—	0.460	0.214	1.134	1.286	0.590[a]	[13]
DMM	—	—	0.452	0.224	1.139	—	0.578[b]	[15]

[a] Exponents determined using the scaling relation $\beta = \delta\nu_\parallel$.
[b] Taken from [18].

In order to gain further insight into the spreading behaviour of the fire, we have evaluated the fractal dimension D_F of the set of sites which has been ignited at least once. D_F is evaluated only (and just) when the fire arrives at the edge of the lattice, i.e. after fire percolation. The average value for the three critical points listed in table 1 is

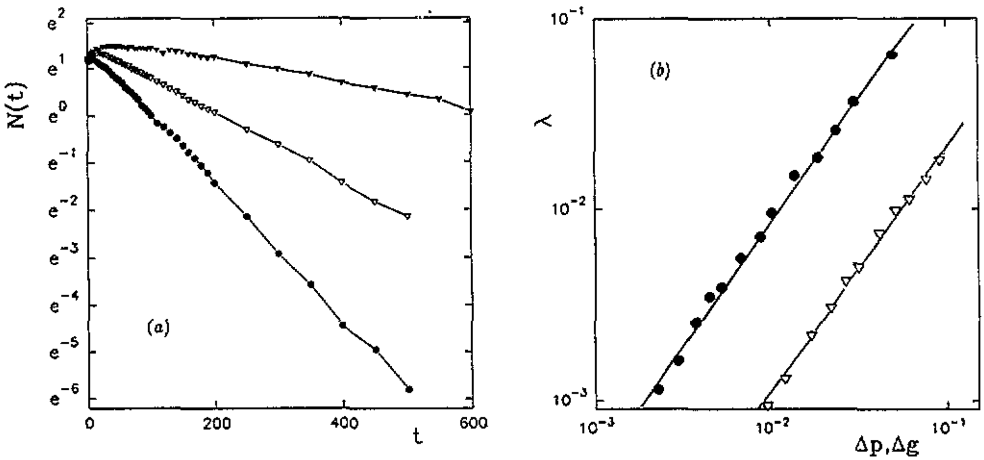

Figure 2. (*a*) Ln–linear plot of $N(t)$ against t according to equation (4), obtained by keeping $p = p_c = 1.0$ constant ($\Delta_2 = 0$) and varying g: $g = 0.5800$ (▼), $g = 0.5850$ (▽) and $g = 0.5900$ (●). λ is evaluated from the asymptotic slopes of the curves. (*b*) Log–log plots of λ against Δ obtained by keeping $p = p_c = 1.0$ constant ($\Delta_2 = 0$) and varying $g(\Delta_1 = \Delta g)$ (●), and keeping $g = g_c = 0.5003$ constant ($\Delta_1 = 0$) and varying $p(\Delta_2 = \Delta p)$ (▽). From the slopes of these curves, the exponents $\nu_\parallel$ listed in table 1 are obtained.

$D_F \cong 1.90 \pm 0.01$. Since $D_F < d = 2$, it follows that the spreading process is not compact and only a fractal set of the forest becomes ignited. Furthermore, the agreement between D_F and the fractal dimension of incipient percolation clusters, i.e. $D_F^{IPC} = 91/48 \cong 1.899$ [1, 2], strongly suggests that this fractal set has standard percolating-like properties.

It should be noted that, for $p = 0$, the FFMIT behaves differently than in the $p \to 0$ limit. In fact, for $p \to 0$, one has that the absorbing state is unique, i.e. the lattice is fully covered by trees. On the contrary, for $p = 0$, since trees can no longer grow, the fire always spreads until it becomes irreversibly extinguished. At the critical point, the fire spreads over the whole lattice for the first time. So, the absorbing state involves two different types of site: trees and empty sites; therefore it is non-unique. This behaviour hinders the spreading study already described because, in the absorbing state, an incipient new fire will only spread over a finite distance. However, one can investigate the spreading of a small fire embedded in an already green lattice, at $t = 0$. Power-law behaviour of equations (1)–(3) are found at $g_c = 0.4655 \pm 0.0005$ and the corresponding exponents are $\eta \cong 0.44 \pm 0.02$, $\delta \cong 0.14 \pm 0.01$ and $z \cong 1.70 \pm 0.02$, which, as expected, depart from the universality class of DP. Also, one has $D_F \cong 1.80 \pm 0.02$, i.e. smaller than D_F^{IPC} but high enough to discard any possible relationship with the dimension of the backbone of the percolation cluster, given by $D_B \cong 1.62 \pm 0.02$ [1, 2].

The validity of equation (4) has to be treated in the subcritical regime, i.e. within the absorbing state with all sites occupied by green trees ($g \geqslant g_c(p_c)$). In fact, the decay constant ($\lambda = \xi_t^{-1}$), governing the long-time behaviour of $N(t)$ in equation (4), behaves according to

$$\lambda = \xi_t^{-1} = \{\mathcal{D}(\Delta_1, \Delta_2)\}^{\nu_\parallel} \tag{5}$$

so, knowing $\{p_c, g_c\}$, one can determine $\nu_\parallel$. Figure 2(*a*) shows ln–linear plots of $N(t)$ against t, where the predicted exponential decay is verified. From the slopes of these curves, λ can be determined and subsequent log–log plots of λ against Δ (figure 2(*b*)) allow us to evaluate $\nu_{\parallel}$, as listed in table 1. The obtained exponents fully agree with the best estimate of $\nu_{\parallel}$ for DP.

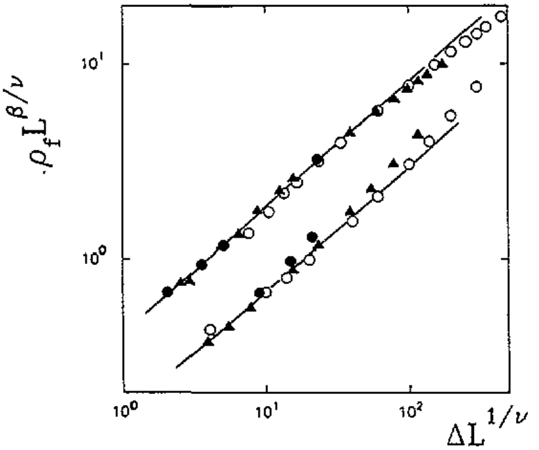

Figure 3. Log–log plots of $\rho_f L^{\beta/\nu}$ against $\Delta L^{1/\nu}$ according to equation (7): $L = 256$ (○), $L = 128$ (▲) and $L = 64$ (●). The upper curve is obtained by keeping $p = p_c = 0.1020$ constant ($\Delta_2 = 0$) and varying $g(\Delta_1 = \Delta g)$, while the lower curve is obtained by varying both p and g with $\Delta_1 = \Delta_2 = \Delta$, $p_c = 0.1020$ and $g_c = 0.5003$.

As in standard second-order transitions, continuous irreversible phase transitions can also be studied using finite-size scaling theory. At criticality, the correlation length ξ_s (space direction) diverges according to

$$\xi_s \propto \{\Gamma(\Delta_1, \Delta_2)\}^{-\nu_\perp} \qquad \Delta_1, \Delta_2 \to 0 \tag{6}$$

where $\nu_\perp$ is the correlation-length exponent in the space direction [18]. Furthermore, the natural order parameter is the fire density (ρ_f), which at criticality takes the following scaling form:

$$\rho_f(p, g, L) = L^{-\beta/\nu_\perp} f\{\Gamma(\Delta_1, \Delta_2) L^{1/\nu_\perp}\} \tag{7}$$

where f is a suitable scaling funciton and β is the order-parameter critical exponent. The exponents β and $\nu_\perp$, necessary to test the scaling ansatz of equation (7), can be determined using scaling relations between already calculated exponents. In fact, the best available values of β are, for example, $\beta \cong 0.590$ [13] (obtained through the scaling relation $\beta = \nu_{\parallel}\delta$ [21]) and $\beta \cong 0.578$ (obtained by means of a direct measurement ([20] and references therein)). Using the scaling relation $\beta = \nu_{\parallel}\delta$ [12], excellent agreement with these figures is obtained for the FFMIT (see table 1). Furthermore, an improved estimate of the correlation-length exponent is given by $\nu_\perp \cong 0.729$ [13]. Using the scaling relation [12] $z = 2\nu_\perp/\nu_{\parallel}$ and the values listed in table 1, one gets $\nu_\perp \cong 0..739$, $\nu_\perp \cong 0.713$ and $\nu_\perp \cong 0.727$ for the FFMIT. So, figure 3 shows excellent data collapsing in log–log plots of $\rho_f L^{\beta/\nu_\perp}$ against $\Delta L^{1/\nu_\perp}$ for lattices of different size and data taken in the supercritical region (i.e. the stationary state). One set of data (the upper curve in figure 3) is obtained keeping p constant ($\Delta_2 = 0, p_c = 1.0$) and varying $\Delta_1 = g - g_c$, with $g_c = 0.5762$, i.e. vertical approach to the critical line $g_c(p_c)$. The slopes of the straight lines obtained by least-square fits of the data are $\beta \cong 0.62 \pm 0.03$ ($L = 256$) and $\beta \cong 0.62 \pm 0.03$ ($L = 128$). The second set of data (lower curve in figure 3) is obtained by 'obliquely' approaching the critical line, i.e. keeping $\Delta_1 = \Delta_2 = \Delta$ with $p_c = 0.1020$ and $g_c = 0.5003$. In this case, the slopes of the straight lines close to the critical point give $\beta \cong 0.58 \pm 0.02$ ($L = 256$) and $\beta \cong 0.63 \pm 0.03$ ($L = 128$). These exponents are in agreement, within error bars, with the already mentioned best available values of β, however, systematic deviations are observed, probably due to metastibilities of the system and corrections to scaling, which are neglected. Collapsing data are poorer in the lower curve of figure 3, presumably due to a systematic deviation from the 'Euclidean' distance between $\{p, g\}$ and $\{p_c, g_c\}$ and the true distance given by $\Gamma(\Delta_1, \Delta_2)$, which becomes more evident for larger Δ-values.

In conclusion, a detailed study of the critical, subcritical and supercritical behaviour of the FFMIT allows us to conclude that it belongs to the same universality class as Reggeon field theory. This result supports the early conjecture [19] that continuous transitions into an absorbing state may belong to the same universality class. Consequently, the FFMIT and some other problems emerging from (apparently) unrelated branches of science, such as particle physics and catalysis, can be understood by means of a unified description. I expect that these results will stimulate further work in order to search for connections between Reggeon field theory and self-organized criticality.

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